Counting Flags in Eulerian Posets: The *cd*-index

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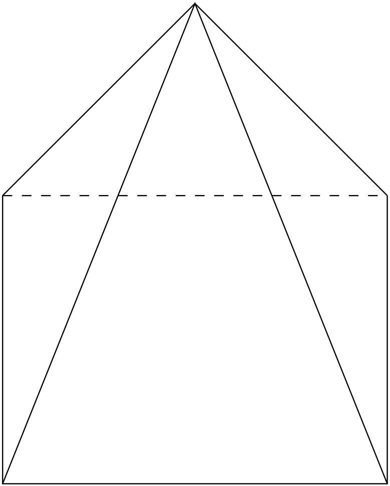
Outline

- Motivation
- Flag vectors
- Eulerian posets
- The *cd* index
- Nonnegativity
- Other inequalities
- Bruhat order
- Toric *h*-vector
- Algebras
- Related parameters

Original motivation

How many faces can a convex polytope have?



Dimension 3

Theorem (Steinitz)

There exists a 3-dimensional polytope with f_0 vertices, f_1 edges and f_2 sides if and only if

$$f_0 - f_1 + f_2 = 2$$

$$f_0 \geq 4 \qquad f_2 \geq 4$$

$$2f_1 \geq 3f_0 \qquad 2f_1 \geq 3f_2$$

For dimension n, the face vector counts the number of faces of all dimensions, 0 through $n - 1$.

We don't have a theorem that characterizes the face vectors of n-dimensional polytopes, even for $n = 4$.

The face vectors of n-dimensional simplicial polytopes have been characterized, by Stanley, and Billera and Lee.

For general polytopes we turn to a vector that encodes more of the combinatorial information of the polytope, the flag vector.

Let $S = \{s_1, s_2, \ldots, s_k\}_< \subseteq \{0, 1, \ldots, n-1\}$

Definition

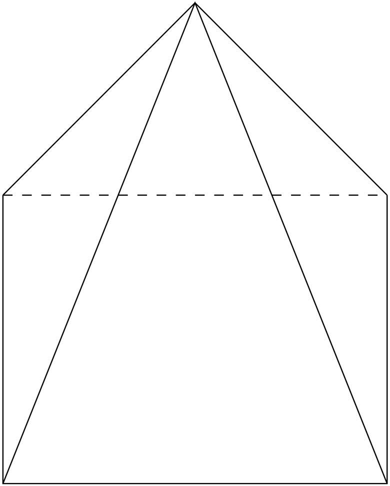
An S-flag of P is a chain

$$\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$$

with $\dim F_i = s_i$

$f_S(P) = \#$ of S-flags of P
$(f_S(P))_{S \subseteq \{0,1,\ldots,n-1\}}$ is the flag vector of P

Example



$f_{\emptyset} = 1$
$f_0 = 5$
$f_1 = 8$
$f_2 = 5$
$f_{01} = 16$
$f_{02} = 16$
$f_{12} = 16$
$f_{012} = 32$

For 3-dimensional polytopes, the flag vector depends linearly on the face vector. This is also true for simplicial polytopes of all dimensions.

We know the complete set of linear equations satisfied by the flag vectors of all n-dimensional polytopes, but the story extends beyond polytopes.

Definition

An Eulerian poset is a graded partially ordered set such that each (nonsingleton) interval $[x, y]$ in the poset has an equal number of elements of even and odd rank.

Examples

- face lattices of convex polytopes
- face posets of regular CW spheres
- intervals in the Bruhat order on finite Coxeter groups
- lattices of regions of oriented matroids

Theorem

The affine dimension of the flag vectors of rank $n+1$ Eulerian posets is $e_n - 1$, where (e_n) is the Fibonacci sequence (with $e_0 = e_1 = 1$). The affine hull of the flag vectors is given by the equations

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup\{j\}}(P) = (1-(-1)^{k-i-1}) f_S(P),$$

where $i \le k-2$, $i,k \in S \cup \{0, n+1\}$, and $S \cap \{i+1, \ldots, k-1\} = \emptyset$.

We need a couple of steps to get from the flag vector to the *cd*-index. First we define the *flag h-vector*:

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P).$$

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Next we write a generating function for the flag h-vector: Associate with $S \subseteq [n]$ the monomial $u_S = u_1 u_2 \cdots u_n$, where $u_i = a$ if $i \notin S$ and $u_i = b$ if $i \in S$. Then write

$$\Psi_P(a, b) = \sum_{S \subseteq [n]} h_S u_S.$$

Example

$f_\emptyset = 1$
$f_0 = 5$
$f_1 = 8$
$f_2 = 5$
$f_{01} = 16$
$f_{02} = 16$
$f_{12} = 16$
$f_{012} = 32$

$h_\emptyset = 1$
$h_0 = 4$
$h_1 = 7$
$h_2 = 4$
$h_{01} = 4$
$h_{02} = 7$
$h_{12} = 4$
$h_{012} = 1$

$$\Psi_P(a,b) = aaa + 4baa + 7aba + 4aab + 4bba + 7bab + 4abb + bbb$$

Jonathan Fine's Inspiration

For P a convex polytope, $\Psi_P(a, b)$ can be written as a polynomial $\Phi_P(c, d)$ with integer coefficients in the noncommuting variables $c = a + b$ and $d = ab + ba$. $\Phi_P(c, d)$ is the *cd*-index of P.

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Some *cd*-indices

n-gon	$cc + (n-2)d$
tetrahedron	$ccc + 2cd + 2dc$
pyramid	$ccc + 3cd + 3dc$
cube	$ccc + 4cd + 6dc$
octahedron	$ccc + 6cd + 4dc$
4-simplex	$cccc + 3ccd + 5cdc + 3dcc + 4dd$
4-cube	$cccc + 6ccd + 16cdc + 14dcc + 20dd$

Important: The number of possible monomials in the *cd*-index of a n-dimensional polytope is exactly the Fibonacci number e_n.

The flag vector is a linear function of the *cd*-index.

So the *cd*-index is an efficient way of encoding the flag vector, incorporating all the linear equations on flag vectors.

Fine believed that the coefficients of the *cd*-index of any convex polytope are all nonnegative. This is true, and in increasing generality.

Theorem

The cd-indices of the following are nonnegative:

- *(Purtill, 1993) low-dimensional polytopes, simple and simplicial polytopes, quasisimplicial polytopes*
- *(Stanley, 1994) S-shellable CW spheres (includes all polytopes)*
- *(Karu, 2006) Gorenstein* posets (posets that are Eulerian and Cohen-Macaulay)*

Nonnegativity in general

Theorem

1. *For the following cd-words w, the coefficient of w as a function of Eulerian posets has greatest lower bound 0 and has no upper bound:*
   - $c^i dc^j$, *with* $\min\{i, j\} \leq 1$,
   - $c^i dcd \cdots cdc^j$ *(at least two d's alternating with c's, i and j unrestricted).*
2. *The coefficient of c^n in the cd-index of every Eulerian poset is* 1.
3. *For all other cd-words w, the coefficient of w as a function of Eulerian posets has neither lower nor upper bound.*

Other inequalities

Theorem (Upper Bound Theorem; Billera and Ehrenborg)

Let P be an n-dimensional polytope with r vertices, and let $C(r,n)$ be the cyclic n-polytope with r vertices. Then

$$\Phi_P(c,d) \leq \Phi_{C(r,n)}(c,d).$$

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Theorem (Lower Bound Theorem; Ehrenborg and Karu)

Let B_n be the Boolean lattice of rank n.

- *If P is a Gorenstein* lattice of rank n, then $\Phi_P(c,d) \geq \Phi_{B_n}(c,d)$.*
- *If P is a Gorenstein* poset, and Q is a subdivision of P, then $\Phi_Q(c,d) \geq \Phi_P(c,d)$.*

Bruhat order

Theorem (Reading, 2004)

The set of cd-indices of Bruhat intervals of rank n spans the affine span of cd-indices of Eulerian posets of rank n.

Billera and Brenti extended the *cd*-index for Bruhat intervals to a nonhomogeneous *cd*-polynomial, and used it to give an explicit computation of the Kazhdan-Lusztig polynomials of the Bruhat intervals for any Coxeter group.

Conjecture (Reading)

Let (W, S) be a Coxeter system, and let $[u, v]$ be an interval in the Bruhat order of W with u of length k and v of length $n + k + 1$. Then the cd-index of $[u, v]$ is coefficientwise less than or equal to the cd-index of a dual stacked n-polytope with $n + k + 1$ facets.
In particular the cd-index of an interval $[1, v]$ is less than or equal to the cd-index of a Boolean lattice.

Detour: The toric *h*-vector

Simplicial Polytopes

face vector $\rightarrow$ *h*-vector

Dehn-Sommerville relations (1920s)

Interpretations of the *h*-vectors (1960s, 1970s):
Shellings, Stanley-Reisner ring, toric varieties

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Rational Polytopes

flag vector $\rightarrow$ toric *h*-vector

So *cd*-index $\rightarrow$ toric *h*-vector

But the toric *h*-vector has much less information than the *cd*-index.

Inequalities on the *cd*-index give inequalities on the toric *h*-vector.

Algebras

One underlying concept

Kalai's convolution:

$$f_S^n * f_T^m(P) = \sum_{\substack{x\in P\\ \rho(x)=n}} f_S^n([\hat{0},x]) f_T^m([x,\hat{1}]) = f^{n+m}_{S\cup\{n\}\cup(T+n)}(P).$$

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Billera & Liu Algebra

There is a graded algebra, with the operation of convolution, generated by $f_\emptyset^j$ (j represents the rank of the poset the flag number $f_\emptyset$ is applied to). The algebra contains a two-sided ideal of elements that vanish for all Eulerian posets. The *cd*-index is identified in the quotient.

Coproducts

Ehrenborg and Readdy:

On posets: $\Delta(P) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} [\hat{0}, x] \otimes [x, \hat{1}]$

On "*ab*-polynomials" (generating functions for flag *h*-vectors)

$$\Delta(u_1 u_2 \cdots u_n) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n$$

These are used to define coalgebras. A Newtonian coalgebra map from the poset colagebra to the *ab*-index coalgebra takes Eulerian posets to *cd*-indices.

Stembridge: The peak subalgebra of the algebra of quasisymmetric functions.

Bergeron, et al.: The Billera-Liu algebra and the Stembridge algebra are dual Hopf algebras.

Aguilar: Generalization to weighted posets and a relative *ab*-index

Karu: *M*-vector analogue

Fine: Conjectured successor to *cd*-index, giving nonnegative "structure coefficients" for product and pyramid.

Related Parameters

Ehrenborg and Readdy: Generalized *cd*-index for **r**-cubical lattices
Ehrenborg: Modified *cd*-index for *k*-Eulerian posets, where every interval of rank *k* is Eulerian.
Ehrenborg, Hetyei, Readdy: *cd*-series for certain infinite posets.
Murai and Yanagawa: Extended *cd*-index for "quasi CW posets".
Grujić and Stojadinović: Analogue of *cd*-index for building sets via Hopf algebra.
Ehrenborg, Goresky and Readdy: Generalized *cd*-indices for posets arising from Whitney stratified manifolds.

Murai and Nevo: Specializing the *cd*-index of certain regular CW spheres to get face vectors of colored simplicial complexes.
Lee: Extension of the toric *h*-vector of a polytope equivalent to the *cd*-index.
Billera and Brenti: Complete *cd*-index for Bruhat intervals.
Dornian: Local *cd*-index, by analogy to Stanley's local *h*-vector.
Dornian, Katz and Tsang: Mixed *cd*-index for strong formal subdivisions of posets.

Major Open Question:

When the *cd*-index is nonnegative, what does it count?

THANK YOU!

To appear in Contemporary Mathematics (AMS)
arxiv:1901.04939